Chapter 1

Robustness of transient behavior

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1.1 Description of the problem

By definition, a system of the form

$$\dot{x}(t) = Ax(t), \quad t \geq 0 \tag{1.1}$$

($A \in \mathbb{K}^{n\times n}$, $\mathbb{K} = \mathbb{R}, \mathbb{C}$) is exponentially stable if and only if there are constants $M \geq 1$, $\beta < 0$ such that

$$\|e^{At}\| \leq Me^{\beta t}, \quad t \geq 0. \tag{1.2}$$

The respective roles of the two constants in this estimate are quite different. The exponent $\beta < 0$ determines the long term behavior of the system, whereas the factor $M \geq 1$ bounds its short term or transient behavior. In applications large transients may be unacceptable. This leads us to the following stricter stability concept.

Definition 1 Let $M \geq 1$, $\beta < 0$. A matrix $A \in \mathbb{K}^{n\times n}$ is called (M, β)-*stable* if (1.2) holds. □

Here $\beta < 0$ and $M \geq 1$ can be chosen in such a way that (M, β)-stability guarantees both an acceptable decay rate and an acceptable transient behavior.

For any $A \in \mathbb{K}^{n\times n}$ let $\gamma(A)$ denote the spectral abscissa of A, i.e. the maximum of the real parts of the eigenvalues of A. It is well known that $\gamma(A) < 0$ implies exponentially stability. More precisely, for every $\beta > \gamma(A)$ there *exists* a constant $M \geq 1$ such that (1.2) is satisfied. However it is unknown how to determine the minimal value of M such that (1.2) holds for a given $\beta \in (\gamma(A), 0)$.

Problem 1:

a) Given $A \in \mathbb{K}^{n\times n}$ and $\beta \in (\gamma(A), 0)$, determine the minimal value $M_\beta(A)$ of $M \geq 1$ for which A is (M, β)-stable.

b) Find easily computable upper and lower bounds for $M_\beta(A)$ and analyze their conservatism.

Associated to this problem is the design problem for linear control systems of the form

$$\dot{x} = Ax + Bu, \tag{1.3}$$

where $(A, B) \in \mathbb{K}^{n\times n} \times \mathbb{K}^{n\times m}$. Assume that a desired transient and stability behavior for the closed loop is prescribed by given constants $M \geq 1, \beta < 0$, then the pair (A, B) is called (M, β)-*stabilizable* (by state feedback), if there exists an $F \in \mathbb{K}^{m\times n}$ such that $A - BF$ is (M, β)-stable.

Problem 2:

a) Given constants $M \geq 1, \beta < 0$, characterize the set of (M,β)-stabilizable pairs $(A,B) \in \mathbb{K}^{n\times n} \times \mathbb{K}^{n\times m}$.

b) Provide a method for the computation of (M,β)-stabilizing feedbacks F for (M,β)-stabilizable pairs (A,B).

In order to account for uncertainties in the model we consider systems described by

$$\dot{x} = A_\Delta x = (A + D\Delta E)x,$$

where $A \in \mathbb{K}^{n\times n}$ is the nominal system matrix, $D \in \mathbb{K}^{n\times \ell}$ and $E \in \mathbb{K}^{q\times n}$ are given structure matrices, and $\Delta \in \mathbb{K}^{\ell\times q}$ is an unknown perturbation matrix for which only a bound of the form $\|\Delta\| \leq \delta$ is assumed to be known.

Problem 3:

a) Given $A \in \mathbb{K}^{n\times n}$, $D \in \mathbb{K}^{n\times \ell}$ and $E \in \mathbb{K}^{q\times n}$, determine analytically the $(M,\beta)-$*stability radius* defined by

$$r_{(M,\beta)}(A; D, E) = \inf\left\{\|\Delta\| \in \mathbb{K}^{\ell\times q},\, \exists \tau > 0 : \, \|e^{(A+D\Delta E)\tau}\| \geq M e^{\beta\tau}\right\}. \tag{1.4}$$

b) Provide an algorithm for the calculation of this quantity.

c) Determine easily computable upper and lower bounds for $r_{(M,\beta)}(A; D, E)$.

The two previous problems can be thought of as steps towards the following final problem.

Problem 4: Given a system $(A,B) \in \mathbb{K}^{n\times n} \times \mathbb{K}^{n\times m}$, a desired transient behavior described by $M \geq 1, \beta < 0$, and matrices $D \in \mathbb{K}^{n\times \ell}$, $E \in \mathbb{K}^{q\times n}$ describing the perturbation structure,

a) characterize the constants $\gamma > 0$ for which there exists a state feedback matrix such that

$$r_{(M,\beta)}(A - BF; D, E) \geq \gamma\,. \tag{1.5}$$

b) Provide a method for the computation of feedback matrices F such that (1.5) is satisfied.

1.2 Motivation and history of the problem

Stability and stabilization are fundamental concepts in linear systems theory and in most design problems exponential stability is the minimal requirement that has to be met. From a practical point of view, however, the transient behavior of a system may be of equal importance and is often one of the criteria which decide on the quality of a controller in applications. As such, the notion of $(M,\beta)-$ stability is related to such classical design criteria as "overshoot" of system responses.

In particular, if linear design is performed as a local design for a nonlinear system large transients may result in a small domain of attraction. The relation between domains of attraction and transient behavior of linearizations at fixed points is an active field motivated by problems in mathematical physics, in particular, fluid dynamics, see [1, 10] and references therein. Related problems occur in the study of iterative methods in numerical analysis, see e.g. [3].

We would like to point out that the problems discussed in this note give *pointwise* conditions in time for the bounds and are therefore different from criteria that can be formulated via integral constraints on the positive time axis. In the literature such integral criteria are sometimes also called bounds on the transient behavior, see e.g. [9] where interesting results are obtained for this case.

Stability radii with respect to asymptotic stability of linear systems were introduced in [5] and there is a considerable body of literature investigating this problem. The questions posed in this note are an extension of the available theory insofar as the transient behavior is neglected in most of the available results on stability radii.

1.3 Available results

For Problem 1 a number of results are available. Estimates of the transient behavior involving either quadratic Lyapunov functions or resolvent inequalities are known but can be quite conservative or intractable. Moreover, for many of the available estimates little is known on their conservatism.

The Hille-Yosida Theorem [8] provides an equivalent description of $(M,\beta)-$stability in terms of the norm of powers of the resolvent of A. Namely, A is (M,β)-stable if and only if for all $n \in \mathbb{N}$ and all $\alpha \in \mathbb{R}$ with $\alpha > \beta$ it holds that

$$\| (\alpha I - A)^{-n} \| \leq \frac{M}{(\alpha - \beta)^n}.$$

While this condition is hard to check there is a classical, easily verifiable, sufficient condition using quadratic Lyapunov functions. Let $\beta \in (\gamma(A), 0)$, if $P > 0$ satisfies the Lyapunov inequality

$$A^* P + PA \leq 2\beta P\,,$$

and has condition number $\kappa(P) := \|P\|\|P^{-1}\| \leq M^2$ then A is (M,β)-stable. The existence of $P > 0$ satisfying these conditions may be posed as an LMI-problem [2]. However, it can be shown that if $\beta < 0$ is given and the spectral bound of A is below β then this method is necessarily conservative, in the sense that the best bound on M obtainable in this way is strictly larger than the minimal bound. Furthermore, experiments show that the gap between these two bounds can be quite large. In this context, note that the problem cannot be solved by LMI techniques since the characterization of the optimal M for given β is not an algebraic problem.

There is a large number of further upper bounds available for $\|e^{At}\|$. These are discussed and compared in detail in [4, 11], see also the references therein. A number of these bounds is also valid in the infinite-dimensional case.

For Problem 2, sufficient conditions are derived in [7] using quadratic Lyapunov functions and LMI techniques. The existence of a feedback F such that

$$P(A - BF) + (A - BF)^* P \leq 2\beta P \quad \text{and} \quad \kappa(P) = \|P\|\|P^{-1}\| \leq M^2\,, \tag{1.6}$$

or, equivalently, the solvability of the associated LMI problem, is characterized in geometric terms. This, however, only provides a sufficient condition under which Problem 2 can be solved. But the LMI problem (1.6) is far from being equivalent to Problem 2.

Concerning Problem 3 differential Riccati equations are used to derive bounds for the $(M,\beta)-$stability radius in [6]. Suppose there exist positive definite Hermitian matrices P^0, Q, R of suitable dimensions such that the differential Riccati equation

$$\dot{P} - (A - \beta I)P - P(A - \beta I)^* - E^* Q E - PDRD^* P = 0 \tag{1.7}$$

$$P(0) = P^0 \tag{1.8}$$

has a solution on $\mathbb{R}_+$ which satisfies

$$\bar{\sigma}(P(t))/\underline{\sigma}(P^0) \leq M^2, \quad t \geq 0.$$

Then the structured $(M,\beta)-$stability radius is at least

$$r_{(M,\beta)}(A; D, E) \geq \sqrt{\underline{\sigma}(Q)\underline{\sigma}(R)}\,, \tag{1.9}$$

where $\bar{\sigma}(X)$ and $\underline{\sigma}(X)$ denote the largest and smallest singular value of X. However, it is unknown how to choose the parameters P^0, Q, R in an optimal way and it is unknown, whether equality can be obtained in (1.9) by an optimal choice of P^0, Q, R.

To the best of our knowledge no results are available dealing with Problem 4.

Bibliography

[1] J.S. Baggett and L.N. Trefethen. Low-dimensional Models of Subcritical Transition to Turbulence. *Physics of Fluids* 9:1043–1053, 1997.

[2] S. Boyd, L. El Ghaoui, E. Feron, and V. Balakrishnan. *Linear Matrix Inequalities in Systems and Control Theory*, volume 15 of *Studies in Applied Mathematics*. SIAM, Philadelphia, 1994.

[3] T. Braconnier and F. Chaitin-Chatelin. Roundoff induces a chaotic behavior for eigensolvers applied on highly nonnormal matrices. Bristeau, M.-O. (ed.) et al., Computational science for the 21st century. Symposium, Tours, France, May 5–7, 1997. Chichester: John Wiley & Sons. 43-52 (1997).

[4] M. I. Gil'. *Stability of Finite and Infinite Dimensional Systems*. Kluwer Academic Publishers, Boston, 1998.

[5] D. Hinrichsen and A. J. Pritchard. Stability radius for structured perturbations and the algebraic Riccati equation. *Systems & Control Letters*, 8:105–113, 1986.

[6] D. Hinrichsen, E. Plischke, and A. J. Pritchard. Liapunov and Riccati equations for practical stability. In *Proc. European Control Conf. ECC-2001*, Porto, Portugal, (CD-ROM), pp. 2883–2888, 2001.

[7] D. Hinrichsen, E. Plischke, and F. Wirth. State Feedback Stabilization with Guaranteed Transient Bounds. To appear in *Proceedings of MTNS-2002*, Notre Dame, IN, USA, August 2002.

[8] A. Pazy. *Semigroups of Linear Operators and Applications to Partial Differential Equations*. Springer-Verlag, New York, 1983.

[9] A. Saberi, A.A. Stoorvogel, and P. Sannuti. *Control of Linear Systems with Regulation and Input Constraints*. Springer-Verlag, London, 2000.

[10] L. N. Trefethen. Pseudospectra of linear operators. *SIAM Review*, 39(3):383–406, 1997.

[11] K. Veselić. Bounds for exponentially stable semigroups. preprint Fernuniversität Hagen, Hagen, Germany, 2001.