Stabilization of Linear Systems with Prescribed Transient Bounds

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Abstract

The feedback synthesis of linear systems satisfying certain exponential bounds on their transient behavior is studied. We present sufficient conditions ensuring that there exists a state feedback matrix such that the closed loop system satisfies a desired exponential growth bound. Special interest is directed to the case of generating contraction semigroups, which under certain conditions can be characterized by a simple growth condition on the kernel of the adjoint of the input matrix. Finally, bounds using quadratic norms are investigated.

1 Introduction

The study of the asymptotic stability of a dynamical system ignores transient effects, which are of importance, e.g. in the presence of physical constraints. We therefore are in need for a notion of stability which takes these transient effects into account. For example, Figure 1 shows the graph of $t \mapsto \left\| e^{At} \right\|$ for a matrix $A \in \mathbb{R}^{7\times 7}$ with spectrum $\sigma(A) = \{-1, -1 \pm 10i, -1 \pm 20i, -1 \pm 25i\}$. Although asymptotically stable, there exists a solution of $\dot{x} = Ax$ for which the norm is amplified by the factor 600 before eventually decaying. Unfortunately, the spectrum contains little information about the location or height of these transient excursions.

The study of transient effects is related to such classical design criteria as "overshoot" of system responses. In [10] a pole-placement technique for SISO discrete-time systems is studied which reduces the overshoot by an optimization involving linear matrix inequality

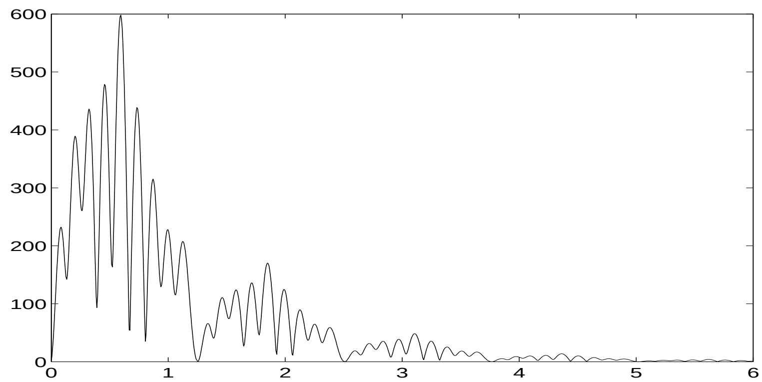

Figure 1: Transient excursions of an asymptotically stable linear system.

constraints. Contraction properties for feedback systems which are bound to modified quadratic Lyapunov functions have been introduced in [8]. [2] use state-space methods to stabilize a continuous-time system by seperating the different time-scales of the system. Transient problems also receive attention in numerical analysis, a chapter in [5] discusses the behaviour of matrix powers, while [9] attacks the problem of numerically solving the matrix exponential. The authors conclude that "any algorithm which tries to pass over the [transient] hump by repeated multiplications is in difficulty" i.e. exploiting the relation $e^{At} = (e^{At/n})^n$ may lead to severe roundoff errors if the transient gain is large.
We proceed as follows. In the next Section 2 a stability concept involving bounds on the transient behavior is introduced and in Section 3 the special case of matrices generating contraction semigroups is discussed. In Section 4 we derive conditions which guarantee the existence of a feedback control such that the solutions of the closed-loop system form a contraction semigroup. In Section 5 this criterion is weakened to account for transient motion satisfying given bounds. In the final two Sections 6 and 7 we study the special case of quadratic norms, which allow for more general statements.

2 Preliminaries

Let $\|\cdot\|$ be an arbitrary vector norm on $\mathbb{C}^n$ or its associated operator norm on $\mathbb{C}^{n\times n}$; the standard Euclidean inner product on $\mathbb{C}^n \times \mathbb{C}^n$ is denoted by $\langle \cdot, \cdot \rangle$. The spectrum of a matrix $A \in \mathbb{C}^{n\times n}$ is denoted by $\sigma(A)$, the spectral abscissa of A is given by $\alpha(A) = \max_{\lambda\in\sigma(A)} \operatorname{Re}\lambda$ and the spectral radius by $\varrho(A) = \max_{\lambda\in\sigma(A)} |\lambda|$. The transposed matrix and the Hermitian adjoint of a matrix A are denoted by $A^\top$ and A^*, respectively. For a Hermitian matrix P, $\lambda_{\max}(P)$ denotes the maximal eigenvalue. The order relation on the set of Hermitian matrices $\mathcal{H}^n$ is defined by $P \succeq Q$ if $P - Q$ is a positive semidefinite matrix. The identity matrix of dimension $n \times n$ is denoted by I_n where the index is dropped when the dimension is clear from the context.
We introduce the following concept of stability and stabilization, where we not only prescribe a decay rate β but also a transient bound M, expanding the notion of exponential

stability.

Definition 1. Suppose $M \geq 1$, $\beta \leq 0$ are given constants. A linear time-invariant system of the form

$$\dot{x}(t) = Ax(t), \quad t \geq 0, \qquad A \in \mathbb{C}^{n\times n} \tag{1}$$

is said to be (M,β)-*stable* with respect to the operator norm $\|\cdot\|$ if it satisfies

$$\left\|e^{At}\right\| \leq Me^{\beta t} \quad \text{ for } t \geq 0. \tag{2}$$

We will call (1) *strictly* (M,β)-stable, if the inequalities in (2) are strict,

$$\left\|e^{At}\right\| < Me^{\beta t} \quad \text{ for } t > 0, \tag{2'}$$

it is called *strongly* (M,β)-stable, if there exists $\beta' < \beta$ such that (1) is (M,β')-stable. A linear time-invariant system of the form

$$\dot{x}(t) = Ax(t) + Bu(t), \quad t \geq 0, \qquad A \in \mathbb{C}^{n\times n}, B \in \mathbb{C}^{m\times n} \tag{3}$$

is said to be *(strictly/strongly)* (M,β)-*stabilizable* by state feedback, if there exists a matrix $F \in \mathbb{C}^{n\times m}$ such that the closed loop system $\dot{x}(t) = (A + BF)x(t)$ is (strictly/strongly) (M,β)-stable.

Any (M,β)-stable system is *exponentially stable* with an *asymptotic growth rate* (or *spectral abscissa*) $\alpha(A) = \max_{\lambda\in\sigma(A)} \operatorname{Re}\lambda \leq \beta$. The special case $M = 1$ is treated in following section.

3 Contractions

Let us first study the case of (M,β)-stability with $M = 1, \beta = 0$. A matrix $A \in \mathbb{C}^{n\times n}$ which is (strictly, strongly) $(M = 1, \beta = 0)$-stable is said to *generate a (strict, strong) contraction semigroup* $T(t) = e^{At}, t \geq 0$. Now, the matrix A generates a contraction semigroup if the closed unit ball $\mathbb{B} = \{x \in \mathbb{C}^n;\ \|x\| \leq 1\}$ is forward invariant under the flow of $\dot{x} = Ax$, i.e. for every $t > 0$ the inclusion $e^{At}\mathbb{B} \subset \mathbb{B}$ holds. Note that this condition needs only to be checked for an infinitesimally small $t > 0$. For this, we only have to study the initial growth rate of $\|T(t)\|$. This growth rate can be described in terms of convex analysis. Recall that the *dual norm* $\|\cdot\|^*$ of an arbitrary vector norm $\|\cdot\|$ is given by

$$\|y\|^* := \max\{\operatorname{Re}\langle y,x\rangle \mid \|x\| \leq 1\} = \max\{|\langle y,x\rangle| \mid \|x\| \leq 1\}.$$

Two vectors $(x,y) \in \mathbb{C}^n \times \mathbb{C}^n$ form a *dual pair*, if $\|y\|^*\|x\| = \langle y,x\rangle \neq 0$. A *normed dual pair* additionally satisfies $\|x\| = 1 = \|y\|^*$. By the Hahn-Banach Theorem the set of dual vectors $\{y \in \mathbb{C}^n |\ (x,y) \text{ dual pair}\}$ and the set of bidual vectors $\{x \in \mathbb{C}^n |\ (x,y) \text{ dual pair}\}$ are never empty.

Definition 2. The initial growth rate with respect to the norm $\|\cdot\|$ is defined by

$$\mu(A) := \sup \left\{ \operatorname{Re} \frac{\langle y, Ax \rangle}{\langle y, x \rangle} \;\middle|\; (x, y) \text{ is a dual pair} \right\}. \tag{4}$$

In the literature this quantity is sometimes called *logarithmic norm* [12] or *matrix measure* [13]. In the following we collect some facts the proof of which can be found in these references. For duality issues, we refer to the discussion of dissipativity and duality in [3].

Proposition 3. *Suppose that $A \in \mathbb{C}^{n\times n}$ and $\|\cdot\|$ is an operator norm on $\mathbb{C}^{n\times n}$. Then the initial growth rate of A is also given by*

$$\begin{aligned}\mu(A) &= \tfrac{d}{dt^+} \left\|e^{At}\right\| \big|_{t=0} = \lim_{t\searrow 0} t^{-1} \log \left\|e^{At}\right\| \\ &= \lim_{t\searrow 0} t^{-1} \left(\|I + At\| - 1\right) = \lim_{r\to\infty} \left(\|rI + A\| - r\right) \\ &= \inf \left\{\mu \in \mathbb{R};\ \forall t \geq 0\ \left\|e^{At}\right\| \leq e^{\mu t}\right\}.\end{aligned}$$

Let us now use (4) to derive a formula for the initial growth rate associated with the Euclidean norm. This norm is self-dual, i.e. all dual pairs (x, y) are of the form $y = x$. Hence

$$\mu_2(A) = \sup_{\|x\|_2=1} \operatorname{Re} \langle x, Ax \rangle = \sup_{x\neq 0} \frac{\langle x, (A + A^*)x \rangle}{2\langle x, x \rangle} = \tfrac{1}{2}\lambda_{\max}(A + A^*). \tag{5}$$

For a given positive definite Hermitian matrix $P \succ 0$ we introduce the weighted inner product $\langle x, y \rangle_P = \langle x, Py \rangle$ and its associated norm $\|\cdot\|_P$ called the *P-norm.* Again, a dual vector of x is uniquely determined, given by $y = Px$. The initial growth rate associated with this P-norm is therefore

$$\mu_P(A) = \sup_{x\neq 0} \frac{\langle x, Ax \rangle_P + \langle Ax, x \rangle_P}{2\langle x, x \rangle_P} = \sup_{x\neq 0} \frac{\langle x, (PA + A^*P)x \rangle}{2\langle x, Px \rangle}. \tag{6}$$

The next proposition gathers some properties of the initial growth rate $\mu(A)$.

Proposition 4. *Suppose that $\mu(\cdot)$ denotes the initial growth rate associated with $\|\cdot\|$. For matrices $A, A' \in \mathbb{C}^{n\times n}$ and scalars $s \in \mathbb{C}$ we have*

$$\begin{aligned}\mu(A + A') \leq \mu(A) + \mu(A'), &\qquad \mu(A + sI) = \mu(A) + \operatorname{Re} s, \\ -\|A\| \leq -\mu(-A) \leq -\alpha(-A), &\qquad \alpha(A) \leq \mu(A) \leq \|A\|\end{aligned}$$

where $\alpha(A)$ is the spectral abscissa.

This proposition allows us to move from arbitrary decay rates $\beta < 0$ to contractions with $\beta = 0$ without loss of generality, as by Proposition 3 for $t \geq 0$

$$\mu(A) \leq \beta \iff \left\|e^{At}\right\| \leq e^{\beta t} \iff \left\|e^{(A-\beta I)t}\right\| \leq 1 \iff \mu(A - \beta I) \leq 0.$$

Especially, the initial growth rate serves as an indicator for contractions.

Corollary 5. *The matrix A generates a contraction semigroup if and only if $\mu(A) \leq 0$. Moreover, A generates a strong contraction semigroup if and only if $\mu(A) < 0$..*

The matrix A is called *dissipative* with respect to a given norm if $\mu(A) \leq 0$. In a Hilbert space context, Corollary 5 is a consequence of a theorem of Lumer and Phillips [14], namely A is dissipative if and only if it generates a contraction semigroup.
Note that there exist generators A of contraction semigroups which satisfy $\left\|e^{At}\right\| < 1$ for $t > 0$ but $\mu(A) = 0$. Consider for example the Jordan block $J = \left(\begin{smallmatrix} -0.5 & 1 \\ 0 & -0.5 \end{smallmatrix}\right)$. Then $\mu(J) = 0$, but for the spectral norm a direct calculation shows $\left\|e^{Jt}\right\| = \frac{1}{2}e^{-1/2t}(\sqrt{4+t^2}+t)$ which is strictly less than 1 for $t > 0$. To avoid such phenomena we will in the following use the notion of strong contractions and strong (M, β)-stability.
Corollary 5 can also be interpreted as stating that the following three items are equivalent.

- $\mu(A) < 0$,
- $\|\cdot\|$ is a (strong) Lyapunov function for $\dot{x} = Ax$ in the sense that $\|\cdot\|$ strictly decays exponentially along all solutions of $\dot{x} = Ax$, $x(0) \neq 0$.
- A generates a strong contraction.

In Example 8 we will see a matrix that generates a strict but not a strong contraction.

4 Synthesis of Contractions

Now that we have a tool ready at hand which determines if a given system is a contraction, we go one step further and ask under which conditions a real linear time-invariant control system of the form

$$\dot{x}(t) = Ax(t) + Bu(t), \qquad A \in \mathbb{R}^{n\times n}, B \in \mathbb{R}^{n\times m} \tag{3}$$

admits a feedback matrix $F \in \mathbb{R}^{m\times n}$ such that the closed loop system matrix $A - BF$ generates a strong contraction. This feedback has to satisfy $\mu(A - BF) < 0$.
Let us study the case $m = 1$, i.e. we consider the system

$$\dot{x} = Ax + bu, \quad \text{with} \quad A \in \mathbb{R}^{n\times n}, b \in \mathbb{R}^n. \tag{3'}$$

We assume a norm of interest $\|\cdot\|$ to be given and we denote its dual norm by $\|\cdot\|^*$. We set

$$\begin{aligned} V^+ &:= \{x \in \mathbb{R}^n |\ y^*b > 0,\ \forall y:\ (x, y) \text{ is a dual pair}\}, \\ V^- &:= \{x \in \mathbb{R}^n |\ y^*b < 0,\ \forall y:\ (x, y) \text{ is a dual pair}\}, \\ V^0 &:= \{x \in \mathbb{R}^n |\ \exists\, y:\ (x, y) \text{ is a dual pair and } y^*b = 0\}. \end{aligned}$$

For our first result the following assumptions are needed.

(A1) V^0 contains a real hyperplane H^0 defined by a suitable vector $h \neq 0$ through

$$H^0 := \{x \in \mathbb{R}^n |\ h^* x = 0\}\,.$$

(A2) For all $x \in H^0$, $\|x\| = 1$ there is a unique vector y_* such that (x, y_*) is a normed dual pair.

We note that (A1) is satisfied for quadractic norms but not for arbitrary norms. In general, given a specific norm only a few vectors will have this property.

Theorem 6. *Consider system* (3') *and the norm* $\|\cdot\|$, *and assume that (A1) and (A2) hold. Then there exists an* $f \in \mathbb{R}^{1\times n}$ *such that* $A - bf$ *generates a strict contraction with respect to* $\|\cdot\|$ *if and only if*

$$y^* A x < 0 \qquad \text{for all } y \in \ker b^* \text{ and } x \in \mathbb{R}^n \text{ such that } (x,y) \text{ is a dual pair.} \tag{7}$$

Proof. If $A - bf$ generates a strict contraction semigroup then equivalently we have for all dual pairs that

$$y^*(A - bf)x < 0.$$

If $y \in \ker b^*$ this implies that

$$y^*(A - bf)x = y^* A x < 0.$$

for all x such that (x, y) is a dual pair. This shows necessity of (7).
We now show the existence of a suitable feedback under the assumption that (7) holds. Let us define the real halfspaces

$$H^+ := \{x \in \mathbb{R}^n |\ h^* x > 0\}\,, \qquad H^- := \{x \in \mathbb{R}^n |\ h^* x < 0\}\,,$$

and assume without loss of generality that $h^* b > 0$. We claim that $V^+ \subset H^+$ and $V^- \subset H^-$. It is clearly sufficient to show the first assertion as $V^+ = -V^-$. First note, that if (x, y) is a dual pair then $y^* b$ is a subgradient of the convex function

$$g : t \mapsto \|x + tb\|$$

at $t = 0$. If $x \in V^+$ this implies that the function g is strictly increasing in $t = 0$ and therefore for all $t \geq 0$. If $x \in V^+ \cap H^-$ then as $h^* b > 0$ we have that $x + t_1 b \in H^0$ for some $t_1 > 0$. By Assumption (A1) there exists a dual vector y of $x + t_1 b$ which satisfies $b^* y = 0$. This contradicts convexity of g combined with the property that g is strictly increasing in t_1. As $V^+ \subset H^+$, $V^- \subset H^-$ we have that $y^* b h^* x \geq 0$ for all dual pairs (x, y) and it is easy to see that $y^* b h^* x > 0$ for all dual pairs (x, y) such that $y \notin \ker b^*$, $x \notin \ker h^*$. We claim that for α sufficiently large we have

$$y^*(A - \alpha b h^*)x < 0$$

for all dual pairs (x, y). Note that it is sufficient to prove this on the compact set

$$Z := \{(x, y) | \, \|y\|^* = \|x\| = \langle y, x\rangle = 1\} .$$

By continuity, the set $Z_- \subset Z$ of points satisfying $y^*Ax < 0$ is open in Z. Assumption (A2) now implies that Z_- contains a set of the form

$$Z_\varepsilon := \{(x, y) \in Z | \, -\varepsilon < h^*x < \varepsilon\}$$

for $\varepsilon > 0$ sufficiently small. Now if $(x, y) \in Z \setminus Z_\varepsilon$, then $|h^*x| \geq \epsilon$. Furthermore there is a $\delta > 0$ so that $y^*Ax \geq 0$ implies $|y^*b| > \delta$, otherwise we obtain a contradiction to (7). Setting

$$\alpha := 2 \frac{\max_{(x,y)\in Z} |y^*Ax|}{\varepsilon\delta} > 0$$

we easily see that

$$y^*Ax - \alpha y^*bh^*x < 0$$

for all $(x, y) \in Z$. □

Remark 7. We note that the construction in the previous proof relies on a high gain type argument. We construct h such that always $y^*bh^*x \geq 0$ and this quantity is even positive for the interesting dual pairs (x, y). This implies that if $A - \alpha_0 bh^*$ generates a strict contraction semigroup the same is true for $A - \alpha bh^*$ for *all* $\alpha \geq \alpha_0$. The following example shows that such a high gain idea is not feasible in all situations. Let

$$A = \begin{pmatrix} -1 & c \\ -2 & d \end{pmatrix}, \qquad b = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

and assume that we want to generate a strict contraction with respect to the 1-norm $\|\cdot\|_1$. The kernel $\ker b^* = \mathbb{R}\binom{1}{0}$ and the (unique) vector x such that $(x, \binom{1}{0})$ is a dual pair is $x = e_1$. An easy calculation shows $(1\,0)A\binom{1}{0} = -1$ so that condition (7) is satisfied. Note that H from (A1) has to be $H = \mathbb{R}e_1$, but (A2) is not satisfied for H. Also Ae_1 is not pointing inside the unit ball of $\|\cdot\|_1$ as can be seen by calculating $(1\ -1)A\binom{1}{0} = 1$ and noting that $(1\ -1)^\top$ is dual to e_1. If we now consider possible feedback matrices $(f_1\, f_2)$ we see

$$A - bf = \begin{pmatrix} -1 & c \\ -2 - f_1 & d - f_2 \end{pmatrix}$$

from which it follows that possible choices for f_1 that lead to a matrix infinitesimally decreasing in e_1,i.e. the first column is diagonally dominant, are $f_1 \in (-3, -1)$. Similarly, $f_2 > \max\{d+c, d-c\}$ ensures that $A - bf$ is pointing inward at e_2. Hence $A - bf$ generates a contraction semigroup with respect to $\|\cdot\|_1$ if and only if

$$f \in \left\{[f_1, f_2] \in \mathbb{R}^{1\times 2} \middle| \, f_1 \in (-3, -1), f_2 > \max\{d + c, d - c\}\right\} .$$

In particular, for any choice of f that generates a strict contraction semigroup there is an α_0 such that for *all* $\alpha \geq \alpha_0$, $A - \alpha bf$ is *not* dissipative.

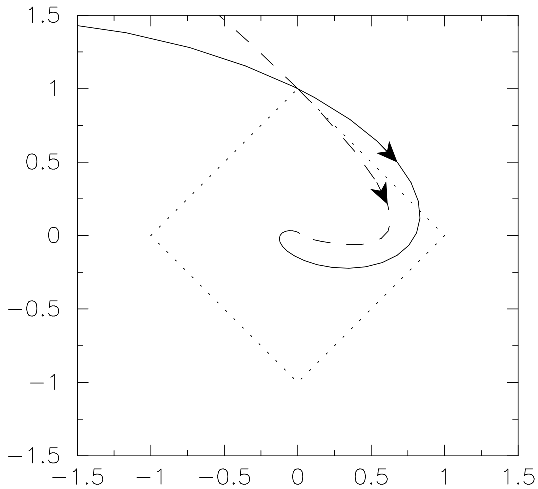

Figure 2: A closed loop contraction with respect to $\|\cdot\|_1$.

Example 8. Choosing $c = 6, d = -3$ in the previous remark gives $A = \left(\begin{smallmatrix} -1 & 6 \\ -2 & -3 \end{smallmatrix}\right)$. The allowed feedback matrices $[f_1, f_2]$ can be selected from $f_1 \in (-3, -1)$ and $f_2 > 3$. Figure 2 shows a trajectory of $\dot{x} = Ax$ which leaves the (dotted) unit box of $\|\cdot\|_1$ and a trajectory of the closed loop system with $f = [-1, 3]$. Here $A - bf = \left(\begin{smallmatrix} -1 & 6 \\ -1 & -6 \end{smallmatrix}\right)$ is only marginally diagonally dominant. The closed loop system generates a strict contraction but not a strong contraction. For $f = [-3, 3]$ the closed loop becomes marginally stable as $A - bf = \left(\begin{smallmatrix} -1 & 6 \\ 1 & -6 \end{smallmatrix}\right)$.

To treat the case of higher dimensional input spaces can be easily obtained from Theorem 6. Again the assumption (A1) and (A2) are crucial. To apply the same arguments as before we have to assume that for each of the columns of B the assumptions $(A1), (A2)$ are satisfied individually. Note, however, that using a state transformation R on the input space, this property might be obtained for the matrix BR, while it is false for B.

Theorem 9. *Consider system* (3) *with $A \in \mathbb{R}^{n\times n}, B \in \mathbb{R}^{n\times m}$ and the norm $\|\cdot\|$. Assume that for each column b_j of B, $j = 1, \dots, m$ the properties (A1) and (A2) are satisfied. Then there exists an F such that $A - BF$ generates a strong contraction if and only if*

$$y^* A x < 0 \qquad \text{for all } y \in \ker B^* \text{ and } x \in \mathbb{R}^n \text{ such that } (x, y) \text{ is a dual pair.} \tag{8}$$

By replacing A with $A - \beta I$ in Theorem 9 we obtain the following result for arbitrary decay rates.

Corollary 10. *Under the assumptions of Theorem 9 there exists a feedback matrix $F \in \mathbb{R}^{m\times n}$ for a given decay rate $\beta < 0$ such that $\mu(A - BF) < \beta$ if and only if all dual pairs (x, y) with $y \in \ker B^*$ satisfy $y^* A x < \beta\, y^* x$.*

5 (M, β)-Stability and (M, β)-Stabilizability

In this section let us consider (M, β)-stability instead of contractions. For arbitrary $M > 1$ we introduce a second norm $\nu(\cdot)$, and relate all contraction properties to this norm. To obtain results for the given norm $\|\cdot\|$ we compare both norms by measuring the deformation of their unit balls.

Definition 11. Suppose $\|\cdot\|, \nu(\cdot)$ are norms on $\mathbb{R}^n$. Then the *eccentricity* of $\nu(\cdot)$ with respect to $\|\cdot\|$ is given by

$$\operatorname{ecc}(\nu) = \operatorname{ecc}_{\|\cdot\|}(\nu) := \frac{\max_{\|x\|=1} \nu(x)}{\min_{\|x\|=1} \nu(x)}. \tag{9}$$

The initial growth rate of A and the eccentricity of a norm $\nu(\cdot)$ provide us with a bound for $\left\|e^{At}\right\|$.

Proposition 12. *Suppose $\nu(\cdot)$ is a norm on $\mathbb{R}^n$ with initial growth rate $\mu_\nu(\cdot)$. Then for all $A \in \mathbb{R}^{n\times n}$,*

$$\left\|e^{At}\right\| \le \operatorname{ecc}(\nu) e^{\mu_\nu(A)t},\ t \ge 0.$$

From Proposition 12 we conclude that A is (M,β) stable with respect to $\|\cdot\|$ if there exists a norm $\nu(\cdot)$ with $\operatorname{ecc}(\nu) \le M$ and $\mu_\nu(A) \le \beta$. On the other hand, for (M,β)-stable systems it is easily seen that the function $\nu(x) = \sup_{t\ge 0} e^{-\beta t}\left\|e^{At}x\right\|$ defines a norm for which $\mu_\nu(A) \le 0$ and $\operatorname{ecc}(\nu) \le M$.
Now, let us fix the norm $\|\cdot\|$. To guarantee (M,β)-stabilization we have to find a norm ν such that $\operatorname{ecc}\nu \le M$, and a feedback matrix F such that $\mu(A+BF) \le \beta$.
As the class of norms is very rich–it even includes an exact bound–let us only consider norms for which eccentricity and initial growth rate are easily computable. Therefore we consider weighted norms, i.e. if $W \in \mathbb{R}^{n\times n}$ is an invertible matrix then we define the new norm $\nu_W(x) = \|Wx\|$. The eccentricity of ν_W is then given by the condition number of W, $\kappa(W) = \|W\|\,\|W^{-1}\| \ge 1$. Its initial growth rate can be described as follows.

Proposition 13. *Given a vector norm $\|\cdot\|$ and a weight matrix $W \in \mathbb{R}^{n\times n}$. Then the initial growth rate with respect to the weighted norm $\|W\cdot\|$ is given by*

$$\mu_{\|W\cdot\|}(A) = \sup_{(x,y)\ \text{dual pair of}\ \|\cdot\|} \frac{y^*WAW^{-1}x}{y^*x} = \mu_{\|\cdot\|}(WAW^{-1}).$$

Proof. We have to identify the set of dual vectors for the weighted norm. But if (x,y) is a dual pair of $\|\cdot\|$ then the pair $(W^{-1}x, W^*y)$ is a dual pair with respect to $\|W\cdot\|$, as $\|W(W^{-1}x)\| = \|x\|$ and $\langle W^*y, W^{-1}x\rangle = \langle y, x\rangle$. Proposition 3 now implies the statement of the proposition. $\square$

For (M,β)-stabilizability we obtain from Theorem 9 the following characterization

Corollary 14. *Consider system* (3) *and the norm $\|\cdot\|$, and assume that (A1) and (A2) hold for each column of B. System 3 is strongly (M,β)-stabilizable if there exists $W \in \mathbb{R}^{n\times n}$ such that $\kappa(W) \le M$ and for all dual pairs (x,y) of $\|\cdot\|$ with $y \in \ker(WB)^*$ the inequality*

$$y^*WAW^{-1}x < \beta y^*x$$

holds.

6 Quadratic (M, β)- Stability

We now specialize the situation by comparing weighted Euclidean norms, i.e. $\|x\|_P = \sqrt{\langle x, Px\rangle}$, with the standard Euclidean norm $\|x\|_2$. The corresponding stability concept is called *quadratic* (M, β)*-stability* [1].

Definition 15. Given the constants $M \geq 1, \beta < 0$, a matrix $A \in \mathbb{R}^{n\times n}$ is called *quadratically* $(M, \beta)-$*stable* if there exists a positive definite Hermitian matrix $P \succ 0$ such that

$$\kappa(P) \leq M^2 \quad \text{and} \quad PA + A^*P \prec 2\beta P$$

where $\kappa(P) = \|P\|_2 \|P^{-1}\|_2$ is the *spectral condition number of* P. A pair $(A, B) \in \mathbb{R}^{n\times n} \times \mathbb{R}^{n\times m}$ is called *quadratically* (M, β)*-stabilizable* if there exists a matrix $F \in \mathbb{R}^{m\times n}$ such that $A - BF$ is quadratically $(M, \beta)-$stable.

If A is quadratically (M, β)-stable then by (6), $\mu_P(A) < \beta$. In this case A is the generator of a semigroup which is strictly contractive with respect to the P-norm. For convenience we denote $PA + A^*P$ by the *Lyapunov operator* $\mathcal{L}_A(P)$. The growth bound of Proposition 12 now takes the following shape.

Proposition 16. *Quadratically* (M, β)*-stable systems are strongly* (M, β)*-stable: If* $A \in \mathbb{R}^{n\times n}$ *satisfies* $\mathcal{L}_A(P) \prec 2\beta P$ *then there exists* $\beta' < \beta$ *with*

$$\left\|e^{At}\right\| \leq \sqrt{\kappa(P)}e^{\beta' t},\ t \geq 0. \tag{10}$$

Proof. A $P-$norm is also a weighted norm, $\|\cdot\|_P = \left\|P^{1/2}\cdot\right\|$ where $P^{1/2}$ is the Hermitian matrix square root of P. Since $\kappa(P^{1/2}) = \kappa(P)^{1/2}$, Proposition 12 implies (10). $\square$

The gap between (M, β)-stability and quadratically (M, β)-stability may be quite large.

Example 17. Consider the matrix $A = \left(\begin{smallmatrix} -1 & 2k\mu \\ 0 & -2\mu-1 \end{smallmatrix}\right)$ for $\mu > 1$ and $k > 0$. The spectral norm of the matrix exponential for a real 2×2 matrix in upper triangular form $A = \left(\begin{smallmatrix} \lambda_1 & \alpha \\ 0 & \lambda_2 \end{smallmatrix}\right)$, $\lambda_1 \neq \lambda_2$, is given by

$$\left\|e^{At}\right\| = \frac{1}{2}\left|e^{\lambda_1 t} - e^{\lambda_2 t}\right| \left(\sqrt{\coth(\tfrac{1}{2}(\lambda_1 - \lambda_2)t)^2 + (\tfrac{\alpha}{\lambda_1 - \lambda_2})^2} + \sqrt{1 + (\tfrac{\alpha}{\lambda_1 - \lambda_2})^2}\right).$$

Here we obtain for $\beta = -1$ and $\lambda_1 = 1, \lambda_2 = -2\mu-1, \alpha = 2k\mu$ the monotonously increasing function

$$\left\|e^{(A-\beta I)t}\right\| = \tfrac{1}{2}(1-e^{-2\mu t})\left(\sqrt{\coth(\mu t)^2 + k^2} + \sqrt{1+k^2}\right) \xrightarrow{t\to\infty} \sqrt{1+k^2} \quad \text{as} \lim_{x\to\infty} \coth(x) = 1.$$

Hence, $M = \sqrt{1+k^2}$ is the smallest possible bound for strict (M, β)-stability with $\beta = -1$. Now let us examine which bound can be obtained using Proposition 16. The strict Lyapunov inequality $PA + A^*P + 2P > 0$ is unsolvable, but there exist matrices $P > 0$ which solve $P(A - \beta I) + (A - \beta I)^*P \leq 0$ for $\beta = -1$. The matrix $P = \left(\begin{smallmatrix} p_1 & p_3 \\ p_3 & p_2 \end{smallmatrix}\right)$ is a solution of this inequality if and only if

$$kp_1 - p_3 = 0, \qquad kp_3 - p_2 < 0.$$

If we fix $p_1 = 1$ then necessarily $p_3 = k$ and $p_2 > k^2$. With this choice P is positive definite. Other solutions are positive scalar multiples of solutions representable in such a manner. The condition number for a 2×2 real matrix $P > 0$ is given by

$$\kappa(P) = \frac{\operatorname{trace} P}{2 \det P} \left(\operatorname{trace} P + \sqrt{(\operatorname{trace} P)^2 - 4 \det P} \right) - 1$$

which follows from representing $\lambda_{\max}/\lambda_{\min}$ in terms of $\lambda_{\max} \cdot \lambda_{\min}$ and $\lambda_{\max} + \lambda_{\min}$. By writing $p_2 = k^2 + \alpha$ we get $\kappa(\alpha) = \frac{1+k^2+\alpha}{2\alpha} \left((1 + k^2 + \alpha) + \sqrt{(1 + k^2 + \alpha)^2 - 4\alpha} \right) - 1$ the minimum of which is attained at $\tilde{\alpha} = 1 + k^2$. Therefore the best bound obtainable by Proposition 16 is $\sqrt{\kappa(\tilde{\alpha})} = k + \sqrt{1 + k^2}$. In this example there is a gap of k between the Lyapunov bound and the minimal bound M. More interestingly, the quotient $\frac{k+\sqrt{1+k^2}}{\sqrt{1+k^2}} \to 2$ as $k \to \infty$.

The main problem of quadratic (M, β) stability is to find a suitable weight $P \succ 0$ such that both M and β are small. If A is diagonalizable then there exists a regular matrix V whose columns consist of eigenvectors of A which are not necessarily of unit length. With the definition of the positive definite matrix $P = (VV^*)^{-1}$ Proposition 16 holds with delay rate equal to the the asymptotic growth rate $\beta' = \alpha(A)$. But then different scale factors of the columns of V lead to different condition numbers of P.

7 Synthesis of Quadratic (M, β)-stable Systems

We now turn to stabilization issues in the quadratic framework. We not only have to determine the feedback matrix but also a weighting matrix for a suitable inner product to achieve a quadratic (M, β)-stable system by feedback. Let us state a spectral version of Theorem 9.

Theorem 18. *Consider the pair $(A, B) \in \mathbb{R}^{n\times n} \times \mathbb{R}^{n\times m}$ and constants $M \geq 1, \beta < 0$. The system $\dot{x} = Ax + Bu$ is quadratically (M, β)-stabilizable if and only if there exists a matrix $P \succ 0$ with $\kappa(P) \leq M^2$ such that*

$$\mathcal{M}_P(\beta) \cap \ker B^* P = \{0\} \tag{11}$$

where the set $\mathcal{M}_P(\beta)$ is given by

$$\mathcal{M}_P(\beta) := \{v \in \mathbb{R}^n;\; v^* \mathcal{L}_A(P) v \geq 2\beta v^* P v\}\,. \tag{12}$$

The set $\mathcal{M}_P(\beta)$ consists of those initial values for which the solutions have an initial growth of at least β, i. e. $\frac{d}{dt} \left\| e^{At} v \right\|_P \Big|_{t=0} \geq \beta \left\| v \right\|_P$. Thus $\mathcal{M}_P(\beta)$ is the subset of the state space where the dynamics of the uncontrolled system has to be modified in order to meet the control aim.

Proof. As the initial growth rate for the $P-$norm is known from (6) and as $\left\| e^{At} \right\| \leq Me^{\beta t} \iff \left\| e^{(A-\beta I)t} \right\| \leq M$, $t \geq 0$, the following equivalences hold for any P-norm

$$\mu_P(A) < \beta \iff \forall x \in \mathbb{R}^n \setminus \{0\} \colon \langle x, (PA + A^* P)x \rangle < 2\beta \langle x, Px \rangle \iff \mathcal{L}_A(P) \prec 2\beta P \tag{13}$$

where the order relation is with respect to the standard inner product. Let us first assume that there exists a suitable $P \succ 0$ with $\kappa(P) \leq M^2$ and $\mathcal{M}_P(\beta) \cap \ker B^*P = \{0\}$. We show that there exists a feedback matrix F such that $\mu(A-BF) < \beta$ by considering $\mathcal{L}_{A-BF}(P)$. Applying a QR-decomposition of B and transforming the data accordingly we have the following partition of suitably dimensioned blocks.

$$A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}, \qquad B = \begin{pmatrix} R \\ 0 \end{pmatrix}, \qquad BF = \begin{pmatrix} G_1 & G_2 \\ 0 & 0 \end{pmatrix}, \qquad P = \begin{pmatrix} P_1 & P_{12} \\ P_{12}^* & P_2 \end{pmatrix}. \tag{14}$$

With respect to this partition the blocks of $\mathcal{L}_{A-BF}(P)$ take the form

$$\begin{aligned} \mathcal{L}_{A-BF}(P)_{11} &= \mathcal{L}_{A_{11}-G_1}(P_1) + P_{12}A_{21} + A_{21}^*P_{12}^* \\ \mathcal{L}_{A-BF}(P)_{12} &= P_1(A_{12}-G_2) + P_{12}A_{22} + (A_{11}-G_1)^*P_{12} + A_{21}^*P_2 \\ \mathcal{L}_{A-BF}(P)_{22} &= \mathcal{L}_{A_{22}}(P_2) + P_{12}^*(A_{12}-G_2) + (A_{12}-G_2)^*P_{12} \end{aligned}$$

The largest subspace which is invariant under changes of the feedback matrix is given by the kernel of B^*P: Since R in (14) is invertible, the kernel of B^*P is spanned by the columns of $\left(\begin{smallmatrix} -P_1^{-1}P_{12} \\ I \end{smallmatrix}\right)$. Here $P_1 \succ 0$ is invertible as it is a principal submatrix of $P \succ 0$. The term

$$\begin{pmatrix} -P_{12}^*P_1^{-1} & I \end{pmatrix} \mathcal{L}_{A-BF}(P) \begin{pmatrix} -P_1^{-1}P_{12} \\ I \end{pmatrix} = \mathcal{L}_{A_{22}-A_{21}P_1^{-1}P_{12}}(P_2 - P_{12}^*P_1^{-1}P_{12}) \tag{15}$$

does not depend on the choice of the feedback matrix. Hence, for every $x \in \ker B^*P$ the term $x^*\mathcal{L}_{A-BF}(P)x$ is independent of F.

To achieve a growth rate of at most β, the inequality $\mathcal{L}_{A-BF}(P) \prec 2\beta P$ has to hold, see (13), which transforms into

$$\begin{pmatrix} I & 0 \\ -P_{12}^*P_1^{-1} & I \end{pmatrix} \mathcal{L}_{A-BF}(P) \begin{pmatrix} I & -P_1^{-1}P_{12} \\ 0 & I \end{pmatrix} \prec 2\beta \begin{pmatrix} P_1 & 0 \\ 0 & P_2 - P_{12}^*P_1^{-1}P_{12} \end{pmatrix}. \tag{16}$$

Using a Schur complement this is equivalent to the following two conditions

$$\begin{aligned} \mathcal{L}_{A_{22}-A_{21}P_1^{-1}P_{12}-\beta I}(P_2 - P_{12}^*P_1^{-1}P_{12}) &\prec 0, \\ \mathcal{L}_{A_{11}-G_1-\beta I}(P_1) + P_{12}A_{21} + A_{21}^*P_{12}^* - K^*\mathcal{L}_{A_{22}-A_{21}P_1^{-1}P_{12}-\beta I}(P_2 - P_{12}^*P_1^{-1}P_{12})^{-1}K &\prec 0 \end{aligned} \tag{17}$$

where K is the upper right block of the left hand side matrix product of (16) given by

$$K = P_1(A_{12} - G_2 - (A_{11}-G_1)P_1^{-1}P_{12}) + P_{12}(A_{22} - A_{21}P_1^{-1}P_{12}) + A_{21}^*(P_2 - P_{12}^*P_1^{-1}P_{12}).$$

The negative definiteness of the first condition is guaranteed by the kernel condition (11) as (15) then implies for all $x \in \mathbb{R}^m$

$$\begin{aligned} x^*\mathcal{L}_{A_{22}-A_{21}P_1^{-1}P_{12}}(P_2 - P_{12}^*P_1^{-1}P_{12})x &< 2\beta x^*(-P_{12}^*P_1^{-1}\ I)P\left(\begin{smallmatrix} -P_1^{-1}P_{12} \\ I \end{smallmatrix}\right)x \\ &= 2\beta x^*(P_2 - P_{12}^*P_1^{-1}P_{12})x. \end{aligned}$$

The second condition may be satisfied by choosing F_1 in such a way that $\mathcal{L}_{A_{11}-G_1-\beta I}(P_1) \prec -(P_{12}A_{21} + A_{21}^* P_{12}^*)$. Conversely, if the pair (A, B) is quadratically (M, β)-stabilizable then by definition there exist $P \succ 0$ with $\kappa(P) \leq M^2$ and a feedback matrix F such that the Lyapunov inequality of (13) holds. Then it also holds on $\ker B^*P$, i.e. $\ker B^*P \cap \mathcal{M}_P(\beta) = \{0\}$. □

From the preceding proof we have the following reformulation of the kernel condition (11).

Corollary 19. *With the notation from Theorem 18 and from the partition* (14) *the kernel condition is equivalent to the negative definiteness of the following Lyapunov matrix,*

$$\mathcal{M}_\beta(P) \cap \ker B^*P = \{0\} \iff \mathcal{L}_{A_{22}-A_{21}P_1^{-1}P_{12}-\beta I}(P_2 - P_{12}^* P_1^{-1} P_{12}) \prec 0.$$

This characterization gives necessary conditions on the inner product P as the matrix $A|_{\ker} := A_{22} - A_{21}P_1^{-1}P_{12}$ has to be stable, $P|_{\ker} := P_2 - P_{12}^* P_1^{-1} P_{12} \succ 0$ has to hold, and $\mathcal{L}_{A|_{\ker}}(P|_{\ker})$ has to be negative definite. To select a weight P one could proceed as follows. Choose P_1 and P_{12} such that $A_{22} - A_{21}P_1^{-1}P_{12}$ is stable. Then choose P_2 in such way that $P_2 - P_{12}^* P_1^{-1} P_{12}$ is positive definite and

$$\mathcal{L}_{A|_{\ker}}(P_2) \prec \mathcal{L}_{A|_{\ker}}(P_{12}^* P_1^{-1} P_{12}). \tag{18}$$

Remark 20. Suppose $\beta < 0$ is fixed. If we choose the feedback matrix F to be of the form $F = \gamma B^*P$ the lognorm condition $\mu_P(A - BF) < \beta < 0$ gives rise to the following *parameterized Riccati inequality*

$$PA + A^*P - 2\gamma PBB^*P - 2\beta P \prec 0.$$

By the previous theorem positive definite solutions $P \succ 0$ exist if and only if the kernel condition (11) holds.

Example 21. Consider the system (3) given by

$$A = \begin{pmatrix} -1 & 0 & 0 & 0 & 0 & 0 & -625 \\ 0 & -1 & -30 & 400 & 0 & 0 & 250 \\ -2 & 0 & -1 & 0 & 0 & 0 & 30 \\ 5 & -1 & 5 & -1 & 0 & 0 & 200 \\ 11 & 1 & 25 & -10 & -1 & 1 & -200 \\ 200 & 0 & 0 & -150 & -100 & -1 & -1000 \\ 1 & 0 & 0 & 0 & 0 & 0 & -1 \end{pmatrix}, B = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}. \tag{19}$$

The transient behavior of e^{At} is plotted in Figure 1, the eigenvalues of A are $-1, -1 \pm 10i, -1 \pm 20i, -1 \pm 25i$. The vector $x = e_7 - e_6$ satisfies $x^*Ax = 998$, hence the system is not a contraction with respect to the spectral norm, and as $x \in \ker B^*$, there does not exist a feedback matrix such that the closed loop system is a contraction. The matrix B is already an upper triangular matrix, hence Corollary 19 is directly applicable. Using the partition (14) A_{22} is already stable, hence let us set $P_{12} = 0$. We therefore need to find a

P_2 such that $\mathfrak{L}_{A_{22}}(P_2) \prec 0$ to ensure that there exists a feedback matrix F such that the closed loop system overshoots a most $\kappa(P)^{1/2}$ where $P_1 = I, P_{12} = 0$. Using a semidefinite program we find a positive definite matrix P_2 with $\kappa(P_2)^{1/2} = 315$, see [6]. Hence there exists a feedback matrix such that the transient excursion of closed loop system stays below 315. And indeed choosing $F = -10B^*$ gives even an excursion below 250, as Figure 3 shows.

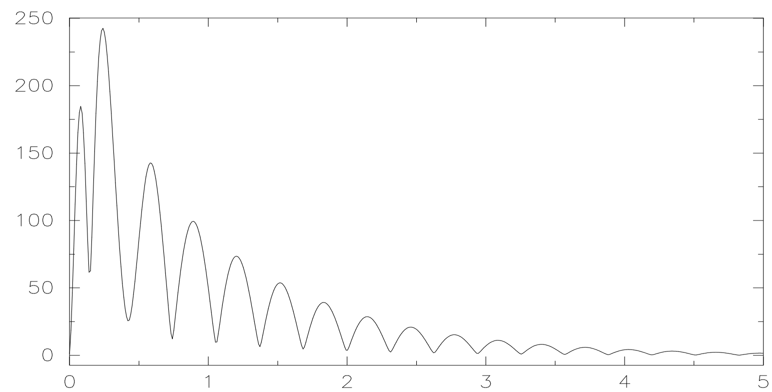

Figure 3: Transient humps of the closed loop system.

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