LIAPUNOV AND RICCATI EQUATIONS FOR PRACTICAL STABILITY

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Abstract

In this note we use Liapunov and Riccati equations to derive bounds for the transient behaviour of a time-invariant linear system. A concept of ”acceptable stability behaviour” is introduced and a sufficient condition for achieving it by time-varying linear state feedback is derived. We also briefly discuss the robustness of acceptable stability behaviour and formulate a corresponding robust control problem.

1 Introduction

Trajectories of a stable linear system may temporarily move a long way from the origin before approaching it as $t \to \infty$. Such a transient behaviour is often exhibited by highly non-normal systems. From a practical point of view, if the “state excursions” are very large the stable system actually behaves like an unstable one. Moreover, if the system is obtained by linearization of a nonlinear system around an equilibrium point the large transients of the linear part may incite the nonlinearities to drive the system permanently far away from the equilibrium. In such cases the practical instability of the equilibrium point is reflected by an extreme thinness of its domain of attraction.

In fluid dynamics the interaction between large transient motions of the linear part and nonlinearities has recently been put forward as an explanation for experimental results where there are observed instabilities of flows at Reynolds numbers which differ significantly from those obtained via spectral analysis, see for example [8].

The union of all the spectra $\sigma(A+\Delta)$ where $\|\Delta\| < \delta$ is called the spectral value set at level δ. The relationship between these sets and transient behaviour has been considered in [3] and an excellent review including some historical remarks can be found in [9].

We first introduce the notion of a transient bound for a given exponential rate and the interplay between the bound and the rate is discussed. Various estimates for the transient bound are obtained via differential and algebraic Liapunov equations. The relationship between the transient bound, stability radii, spectral value sets and algebraic Riccati equations is explored and the possibility of improving the bound by state feedback is discussed. Finally perturbation bounds are derived from differential Riccati equations which ensure acceptable transient behaviour for both linear and nonlinear uncertain systems.

2 General definitions and problem motivation

Throughout this note it is assumed that $A \in \mathbb{K}^{n\times n}$, $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$, $\mathbb{K}^n$ is provided with the standard Euclidean norm $\|\cdot\|$ and $\mathbb{K}^{n\times n}$ with the spectral norm which we also denote by $\|\cdot\|$. The spectrum of A is denoted by $\sigma(A)$ and the largest (resp. smallest) singular value of A by $\overline{\sigma}(A)$ (resp. $\underline{\sigma}(A)$).

Let $b(A) = \max\{\mathrm{Re}\ \lambda;\ \lambda \in \sigma(A)\}$ be the spectral bound of A, then we know that for every $\beta > b(A)$, there exists an M (depending on β), such that

$$\|e^{At}\| \le Me^{\beta t}, \quad t \ge 0. \tag{1}$$

As a consequence the *growth rate* of $(e^{At})_{t\ge 0}$ as $t \to \infty$ is given by the spectral bound

$$\omega(A) := \lim_{t\to\infty} \frac{\ln \|e^{At}\|}{t} = b(A).$$

The spectral bound $b(A)$ governs the long term behaviour of the system $\dot{x} = Ax$, whereas its transient behaviour is reflected in the various values of M which satisfy the inequality (1).

Definition 1. Given any $\beta \ge b(A)$ the *transient bound* of $(e^{At})_{t\ge 0}$ for the exponential rate β is defined to be

$$M_\beta(A) = \inf\{M \in \mathbb{R};\ \forall t \ge 0 :\ \|e^{At}\| \le Me^{\beta t}\}. \tag{2}$$

We set $M_\beta(A) = \infty$ if there is no M such that (1) holds.

In the following we give an example of a stable linear system with large transient motions and show that the inclusion of a nonlinear term results in an equilibrium point at the origin with a thin region of asymptotic stability.

Example 2. Suppose A is a stable, real, upper triangular matrix, $A = \begin{bmatrix} -0.6 & c \\ 0 & -1 \end{bmatrix}$, with $c \in \mathbb{R}$. Then its matrix exponential is given by

$$e^{At} = \begin{bmatrix} e^{-.6t} & 2.5c(e^{-.6t} - e^{-t}) \\ 0 & e^{-t} \end{bmatrix}.$$

Figure 1 shows $(\|e^{At}\|)_{t\ge 0}$ for various values of c using the spectral norm. One sees that the transient bound $M_0(A)$ increases as c

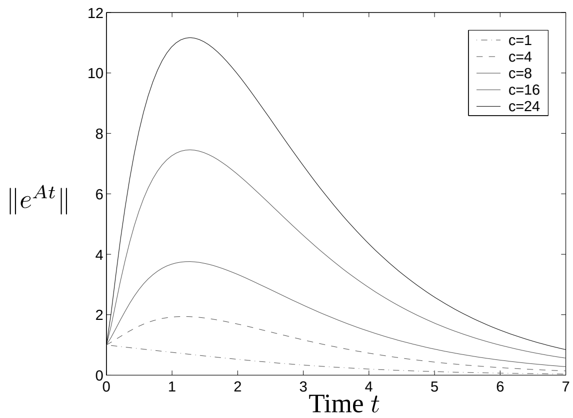

Figure 1: The function $\|e^{A\,\cdot}\|$ for various c

increases and the time at which the maximum is achieved is almost constant. The trajectories in the phase plane from the initial point $(0, 1)$ are shown in the figures below for $c = 8$ and 24, for which $M_0(A) = 3.76$ and 11.2 respectively. The large transient motions are clearly visible. The thin straight lines in the two pictures of Figure

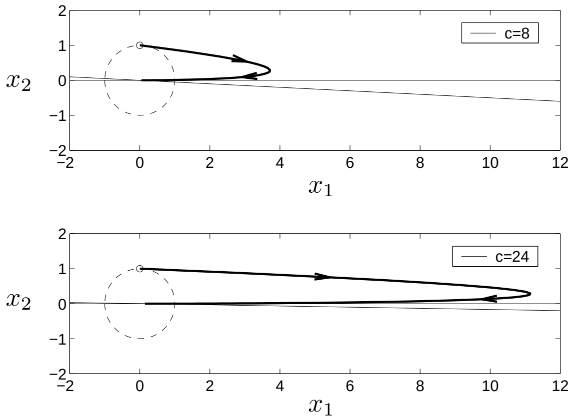

Figure 2: Trajectories for $c = 8, 24$

2 represent contracting eigenmotions of the system. Note that these lines get closer together as c increases; i.e. the eigenvectors become more aligned whilst the eigenvalues remain apart.

Now consider the nonlinear system

$$\dot{x}_1 = x_1(x_1 - 0.6) + cx_2, \qquad \dot{x}_2 = -x_2. \tag{3}$$

This system has a stable equilibrium point at at $(0, 0)$ and an unstable one at $(0.6, 0)$. The phase-portrait is shown in Figure 3 for $c = 8$. Because of the large transient motions of the linearized system one might expect a very thin region of asymptotic stability in the region $\{(x_1, x_2); x_1 \leq 0.6,\, x_2 \geq 0\}$. This is clearly shown in Figure 3 where one should note that the scale in the x_2 direction is five times smaller than that in the x_1 direction. $\square$

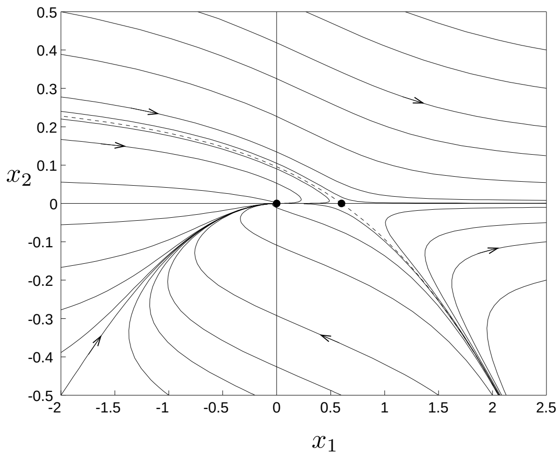

Figure 3: Phase portrait of (3)

3 Transient bounds

It is clear that

$$b(A) \leq \beta \leq \beta' \quad \Rightarrow \quad M_\beta(A) \geq M_{\beta'}(A).$$

and if $\beta > b(A)$, then necessarily $M_\beta(A) < \infty$. However it is possible that $M_\beta(A) \to \infty$ as $\beta \to b(A)$. This will happen if and only if $M_{b(A)}(A) = \infty$ or, equivalently, if and only if the Jordan canonical form of A contains blocks of order ≥ 2 corresponding to eigenvalues $\lambda \in \sigma(A)$ with $\mathrm{Re}\ \lambda = b(A)$.

Obviously $\beta \geq b(A)$ if and only if $b(A_\beta) \leq 0$ for $A_\beta := A - \beta I_n$, and it follows from (2), that

$$M_\beta(A) = M_0(A_\beta), \quad \beta \geq b(A). \tag{4}$$

We now turn to the problem of determining $M_\beta(A)$ for given $A \in \mathbb{K}^{n\times n}$ and $\beta > b(A)$. Since $\lim_{t\to\infty} \|e^{A_\beta t}\| = 0$ there exists $t_\beta \geq 0$ (not necessarily unique) such that

$$\|e^{At_\beta}\| = M_\beta(A)e^{\beta t_\beta}.$$

It follows that if A is stable and $\beta \in (b(A), 0)$, then

$$M_\beta(A)e^{\beta t_\beta} = \|e^{At_\beta}\| \leq M_0(A) = \|e^{At_0}\| \leq M_\beta(A)e^{\beta t_0}$$

and hence

$$M_0(A)e^{-\beta t_0} \leq M_\beta(A) \leq M_0(A)e^{-\beta t_\beta}. \tag{5}$$

Estimates for $M_\beta(A)$ can be obtained via the solution of differential Liapunov inequalities.

Proposition 3. *Suppose $\beta > b(A)$ and $A_\beta = A - \beta I_n$. Given an Hermitian matrix $P^o \in \mathcal{H}_n$, $P^o \succ 0$ then if $P_\beta(\cdot)$ is the solution of the initial value problem*

$$\dot{P} - A_\beta^* P - PA_\beta \succeq 0, \quad P(0) = P^o, \tag{6}$$

we have

$$\sup_{t\geq 0} \overline{\sigma}(P_\beta(t))/\underline{\sigma}(P^o) \geq M_\beta(A)^2.$$

For the special case $P^o = I_n$, the solution of the equation in (6) *satisfies* $\sup_{t\geq 0} \overline{\sigma}(P_\beta(t)) = M_\beta(A)^2$.

If for some $Q \in \mathcal{H}_n$, $Q \succeq 0$

$$\dot{P} - A_\beta^* P - P A_\beta - Q = 0, \quad P(0) = P^o,$$

then

$$P_\beta(t) = e^{A_\beta^* t} P^o e^{A_\beta t} + \int_0^t e^{A_\beta^*(t-s)} Q e^{A_\beta(t-s)} ds.$$

So $\overline{\sigma}(P_\beta(t))$ will be uniformly bounded for $t \geq 0$ if $\beta > b(A)$ and the smallest bound is obtained for $Q = 0$. This suggests that we should have restricted our considerations to the equality in (6). We have chosen not to do so because this precludes the possibility of constant solutions. Indeed since $\sigma(A_\beta) \subset \mathbb{C}_-$, there exists a $P^o \succeq 0$, such that (6) has the constant solution $P_\beta(t) = P_\beta = P^o$. Then if $P_\beta \succ 0$ we obtain the bound

$$\overline{\sigma}(P_\beta)/\underline{\sigma}(P_\beta) \geq M_\beta(A)^2. \tag{7}$$

So an interesting question is: How should one choose $C \in \mathbb{K}^{p\times n}$ with (A, C) observable, such that for the unique solution P_β of the Liapunov equation

$$A_\beta^* P_\beta + P_\beta A_\beta + C^* C = 0 \tag{8}$$

the expression $\overline{\sigma}(P_\beta)/\underline{\sigma}(P_\beta)$ is minimized. The ratio $\overline{\sigma}(P_\beta)/\underline{\sigma}(P_\beta)$ is the *condition number* of P_β, $\kappa(P_\beta)$ and in [6] the following result was proved.

Lemma 4. *Let $\beta > b(A)$. If $P_\beta \succ 0$ and $Q = C^*C \succeq 0$ satisfy* (8)*, then*

$$\|e^{A_\beta t}\| \leq \sqrt{\kappa(P_\beta)} e^{-t[2\lambda_{\max}(P_\beta, Q)]^{-1}}, \quad t \geq 0.$$

where $\lambda_{\max}(P_\beta, Q) = \max_{x\neq 0} \langle x, P_\beta x\rangle \langle x, Qx\rangle^{-1}$ is the maximal eigenvalue of the matrix pencil $\lambda Q - P_\beta$.

If $A_\beta + A_\beta^* = A + A^* - 2\beta I_n \preceq 0$, then we may choose $P_\beta = I_n$ and hence obtain the optimal estimate $\kappa(P_\beta) = \kappa(I_n) = M_\beta(A) = 1$. In fact $\nu(A) = \frac{1}{2}\lambda_{\max}(A + A^*)$ is called the *log-norm* of A [7], and if $\nu(A) \leq 0$, we have

$$\|e^{At}\| \leq e^{\nu(A)t} \leq 1, \quad t \geq 0.$$

If $\nu(A) > 0$ we will see that one can improve (i.e. decrease) the condition number by changing Q to be singular and for such Q the estimate of $\|e^{A_\beta t}\|$ in the lemma is $\sqrt{\kappa(P_\beta)}$ which is the same as that in (7). Since there is no loss of generality in taking $\beta = 0$, we only consider the Liapunov equation

$$A^* P + PA + Q = 0. \tag{9}$$

We will use the following lemma.

Lemma 5. *Suppose $P_1 \succ 0$, $P_2 \succ 0$. Then*

$$\kappa(P_2) < \kappa(P_1) \quad \Longrightarrow \quad \kappa(P_2 + P_1) < \kappa(P_1). \tag{10}$$

As $\kappa(\alpha P) = \kappa(P)$ for $\alpha > 0$, equation (10) also holds for all positive linear combinations $\alpha_1 P_1 + \alpha_2 P_2$, $\alpha_1 > 0, \alpha_2 > 0$. Suppose that we have obtained an estimate for $M_0(A)$ based on an Hermitian pair (P_1, Q_1) satisfying (9), the following proposition uses the lemma to suggest a possible way of improving the estimate.

Proposition 6. *Suppose A is stable and the Hermitian pairs (P_1, Q_1), (P_2, Q_2) satisfy* (9) *with $P_1 \succ 0$, $P_2 \succ 0$, $\kappa(P_2) < \kappa(P_1)$, $Q_1 \succeq 0$, $Q_1 + Q_2 \succeq 0$. Then*

$$M_0(A) \leq \sqrt{\kappa(P_2 + P_1)} < \sqrt{\kappa(P_1)}.$$

In [4] it was noted that the choice $P_2 = \lambda I_n$ (with an apropriate scale factor λ) reduces the condition number, $\kappa(P_1 + \lambda I_n) < \kappa(P_1)$. Namely, suppose that $\nu(A) > 0$ and $Q_1 \succ 0$, then for $P_2 = \lambda I_n, \lambda > 0$, we have $Q_1 + Q_2 = Q_1 - \lambda(A + A^*)$. Then since $\kappa(P_2) = 1$ and $\kappa(P_1) > 1$ if $\nu(A) > 0$, ($P_1 = \alpha I_n$ is not a feasable solution of (9)), the estimate for $M_0(A)$ is always improved provided $Q_1 - \lambda(A + A^*) \succeq 0$. Hence one should choose λ to be the smallest positive eigenvalue of the matrix pencil $Q_1 - \lambda(A + A^*)$. This makes $Q_1 - \lambda(A + A^*)$ singular. The following proposition shows that an optimal Q always exists.

Proposition 7. *Let $A \in \mathbb{C}^{n\times n}$ be a stable matrix. Then there exists a positive semi-definite Hermitian matrix $Q = C^*C$ with $\operatorname{rank} C \leq n$ and (A, C) observable such that the solution P of* (9) *has a minimal condition number $\kappa(P)$ amongst all Hermitian P satisfying $A^*P + PA \preceq 0$.*

Estimates for the transient bound can also be obtained from Kreiss's theorem [10]. Now it may be that we are only interested in the transient behaviour in certain directions, e.g. if a perturbation is known to affect only certain coordinates of the state vector. We take this into account by introducing structure matrices $(B_1, C_1) \in \mathbb{K}^{n\times \ell} \times \mathbb{K}^{q\times n}$ and consider $M_0(A; B_1, C_1) = \sup_{t\geq 0} \|C_1 e^{At} B_1\|$. For this extension Kreiss's theorem takes the form

Theorem 8. *Suppose $A \in \mathbb{K}^{n\times n}$ is a stable matrix and $(B_1, C_1) \in \mathbb{K}^{n\times \ell} \times \mathbb{K}^{q\times n}$ are given structure matrices. Define $R(A; B_1, C_1) := \sup_{\mathrm{Re}\ s>0} (\mathrm{Re}\ s) \|C_1(sI - A)^{-1} B_1\|$. Then*

$$R(A; B_1, C_1) \leq M_0(A; B_1, C_1) \leq (e\, n)\, R(A; B_1, C_1) \tag{11}$$

where $e = \exp(1) = 2.718\ldots$

The bounds can be expressed in terms of the stability radius,

$$r^-(A; B_1, C_1) = \left(\sup_{\omega\in\mathbb{R}} \|C_1(\imath\omega I_n - A)^{-1} B_1\| \right)^{-1}.$$

Suppose $s = \alpha + \imath\omega$ with α, ω real, then

$$\begin{aligned} R(A; B_1, C_1) &= \sup_{\alpha>0} \alpha \sup_{\omega\in\mathbb{R}} \|C_1(\imath\omega I_n - (A - \alpha I_n))^{-1} B_1\| \\ &= \sup_{\alpha>0} \alpha\, r^-(A_\alpha; B_1, C_1)^{-1}, \end{aligned}$$

where $A_\alpha = A - \alpha I_n$. Now the stability radius itself can be characterized via parameterized Riccati equations [2]. Let

$$A_\alpha^* X + X A_\alpha - \rho^2 C_1^* C_1 - X B_1 B_1^* X = 0.$$

Suppose $\rho(\alpha)$ is the supremal value of ρ such that the Riccati equation has Hermitian solutions, then $\rho(\alpha) =$

$r^-(A_\alpha; B_1, C_1)$. Thus $R(A; B_1, C_1) = \sup_{\alpha>0} \alpha/\rho(\alpha)$.
We can also interpret $R(A; B_1, C_1)$ in terms of the associated spectral value sets:

$$\sigma_{\mathbb{C}}(A; B_1, C_1; \delta) = \bigcup_{\Delta \in \mathbb{C}^{\ell \times q},\ \|\Delta\| < \delta} \sigma(A + B_1 \Delta C_1), \quad \delta > 0.$$

$\sigma_{\mathbb{C}}(A; B_1, C_1; \delta)$ is the set of all $\lambda \in \mathbb{C}$ to which at least one eigenvalue of $A + B_1\Delta C_1$ can be moved by a perturbation $\Delta \in \mathbb{C}^{\ell \times q}$ of norm $\|\Delta\| < \delta$. In [1] the following fact was proved

$$\sigma_{\mathbb{C}}(A; B_1, C_1; \delta) = \{ s \in \mathbb{C} ; \|C_1(sI - A)^{-1}B_1\| > \delta^{-1} \}$$

where $\|C_1(sI - A)^{-1}B_1\| := \infty$ for $s \in \sigma(A)$. Now let

$$b_\delta(A; B_1, C_1) = \sup\{ \operatorname{Re} s ; s \in \sigma_{\mathbb{C}}(A; B_1, C_1; \delta)\}.$$

be the *pseudospectral abscissa* at level $\delta > 0$. Then

$$R(A; B_1, C_1) = \sup_{\delta > 0} b_\delta(A; B_1, C_1)/\delta.$$

So if A is stable and for small δ the spectral value sets $\sigma_{\mathbb{C}}(A; B_1, C_1; \delta)$ move deep into the right half plane, then some trajectories of the system $\dot{x} = Ax$ will make large transient excursions.

Example 9. We will compute $R(A; I_n, I_n)$ for the matrices

$$A = \begin{pmatrix} -0.8 & -2 & -8 & 0 \\ 4 & -0.8 & 8 & 0 \\ 0 & 0 & -3.2 & 2 \\ 0 & 0 & 0 & -0.8 \end{pmatrix}, \qquad B_1 = C_1 = I_n.$$

In Figure 4 we have plotted the spectral contours $\|(sI - A)^{-1}\| = \delta^{-1}$ for values of δ equal to .2, .33, .66 and from then in steps of .5 from 1 to 3.

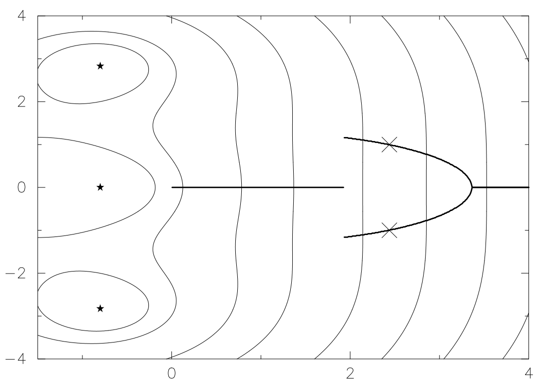

Figure 4: Spectral contours and front locus

Denote by $s(\delta)$ the rightmost point(s) of the contours and let us call the set $\mathcal{F} = \{ s(\delta) ; \delta > 0\} \cap \mathbb{C}_+$ the *front locus* of A. This set is also plotted in the figure as thick lines. We see for small δ that $s(\delta)$ is single valued and real. There is then a discontinuity when $s(\delta)$ takes on two values. From this value of δ it follows a parabolic-like path until it becomes single valued again. The $\times$ marks are those points $s(\hat{\delta})$ where $b_{\hat{\delta}}(A; I_n, I_n)/\hat{\delta} = R(A; I_n, I_n) = 1.43$. Note that only the real part of $s(\hat{\delta})$ is used in the computation of $R(A; I_n, I_n)$. This value is to be compared with $M_0(A) = \sup_{t \geq 0} \|\exp(At)\| = 2.51$. □

4 Achieving acceptable stability behaviour by state feedback

We have seen in the introduction that, from a practical point of view, an asymptotically stable linear system may behave like an unstable one because its dynamics allow for large transient deviations from the origin. Such practical instability cannot be prevented by imposing spectral constraints alone. Instead we will impose the following *practical stability* requirement in time domain:

$$\|e^{At}\| \leq Me^{\beta t}, \quad t \geq 0 \tag{12}$$

where $\beta < 0$ is given. We then have the following:

Stabilization problem. *Given a control system*

$$\dot{x} = Ax + B_2 u, \qquad (A, B_2) \in \mathbb{K}^{n \times n} \times \mathbb{K}^{n \times m} \tag{13}$$

which does not satisfy the stability requirement (12)*, under which conditions is there a feedback law* $u = Fx$, $F \in \mathbb{K}^{m \times n}$ *such that the closed loop system with matrix* $A_F = A + B_2 F$ *satisfies the condition* (12).

In fact we will study the time-varying version of this problem. Suppose we have a piecewise continuous time-varying feedback matrix $F : \mathbb{R} \to \mathbb{K}^{m \times n}$ then the corresponding closed loop system is described by

$$\dot{x} = (A + B_2 F(t))x, \qquad x(0) = x_0 \in \mathbb{K}^n. \tag{14}$$

If $\varphi_F(\cdot, \cdot)$ is the evolution operator generated by $A + B_2 F(t)$ the time-varying version of the practical stability requirement (12) has the form

$$\|\varphi_F(t, 0)\| \leq Me^{\beta t}, \quad t \geq 0. \tag{15}$$

We assume, in the following, that the system has already been stabilized by constant state feedback, so that $b(A) < \beta < 0$, and we now seek to ensure (15).

Lemma 10. *Assume* $\beta > b(A)$*, so* $b(A_\beta) = b(A) - \beta < 0$*. Suppose* $R \in \mathcal{H}_n$, $R \succ 0$ *and consider the differential Liapunov equation*

$$\dot{Z} - A_\beta Z - Z A_\beta^* + B_2 R B_2^* = 0,\ t \geq 0. \tag{16}$$

The unique solution Z_β *of* (16) *on* $\mathbb{R}_+$ *with initial value* $Z_\beta(0) = I_n$ *is given by*

$$Z_\beta(t) = e^{A_\beta t} \left[I_n - \int_0^t e^{-A_\beta s} B_2 R B_2^* e^{-A_\beta^* s} ds \right] e^{A_\beta^* t}. \tag{17}$$

Let $I^+ = [\,0, t_+) = \{t \geq 0;\, Z_\beta(t) \succ 0\}$, then $0 < t_+ \leq \infty$ and the initial value problem

$$\dot{X} + A_\beta^* X + X A_\beta - X B_2 R B_2^* X = 0,\; X(0) = I_n. \quad (18)$$

has the solution $P_\beta(t) = Z_\beta(t)^{-1}$ on I^+. Moreover $\lim_{t\to t_+} \|P_\beta(t)\| = \infty$.
If $F(t) = -RB_2^* P_\beta(t)$, $t \in I^+$, then

$$\varphi_F(t,0)\varphi_F(t,0)^* \preceq Z_\beta(t), \;\; t \in I^+. \quad (19)$$

We see from (17) and (19) that

$$\|e^{A_\beta t}\|^2 - \|\varphi_F(t,0)\|^2 \geq \|e^{A_\beta t}\|^2 - \|Z_\beta(t)\| \geq \underline{\sigma}\left(\int_0^t e^{A_\beta s} B_2 R B_2^* e^{A_\beta^* s} ds\right), \quad t \in I^+.$$

This suggests a possible way of achieving (15).

Proposition 11. *Suppose that $b(A) < \beta < 0$ and there exists a $T \in (0, t_+)$ such that*

$$\|Z_\beta(T)\| \leq M^2/M_\beta(A)^2, \;\; \|Z_\beta(t)\| \leq M^2, \; t \in [0,T] \quad (20)$$

where $Z_\beta(\cdot)$ is as in Lemma 10. Then, choosing as feedback matrix

$$F(t) = \begin{cases} -RB_2^* P_\beta(t) & \text{if } t \in [0,T] \\ 0 & \text{if } t > T \end{cases}$$

where $P_\beta(\cdot)$ is as in Lemma 10, ensures the closed loop system (14) *satisfies* (15).

In order to check whether or not the conditions in the above proposition hold, one needs to compute the solution of the differential Liapunov equation (16) with initial value $Z_\beta(0) = I_n$ and monitor whether or not it is possible to find T such that $\|Z_\beta(T)\| \leq M^2/M_\beta(A)^2$ and on the interval $[0,T]$, $Z_\beta(t)$ does not transcend the value M^2 and $\underline{\sigma}(Z_\beta(t)) > 0$.
We will now give an interpretation of the solution $P_\beta(t)$ of the initial value problem (18) in terms of the following finite time optimal control problem:

Minimize

$$J(x^0, u) = \langle x(T), P_\beta(T)x(T)\rangle + \int_0^T \langle u(t), R^{-1}u(t)\rangle dt$$

subject to

$$\dot{x} = A_\beta x + B_2 u, \quad x(0) = x^0.$$

Let $F(t) = -RB_2^* P_\beta(t), t \in [0,T]$ then we have for every control $u(\cdot) \in L^2(0,T;\mathbb{K}^m)$ and $t \in [0,T]$,

$$\begin{aligned}\frac{d}{dt}\langle x(t), P_\beta(t)x(t)\rangle &= \langle x(t), (\dot{P}_\beta(t) + A_\beta^* P_\beta(t) + P_\beta(t)A_\beta)x(t)\rangle \\ &\quad + 2\,\mathrm{Re}\,\langle x(t), P_\beta(t)B_2 u(t)\rangle \\ &= \langle u(t) - F(t)x(t), R^{-1}(u(t) - F(t)x(t))\rangle - \langle u(t), R^{-1}u(t)\rangle.\end{aligned}$$

Hence integrating from 0 to T, yields

$$J(x^0, u) = \langle x^0, P_\beta(0)x^0\rangle + \int_0^T \langle u(t) - F(t)x(t), R^{-1}(u(t) - F(t)x(t))\rangle dt.$$

So the control $u(t) = -RB_2^* P_\beta(t)x(t)$ minimizes the cost functional $J(x^0,u)$ and the optimal cost is $\|x^0\|^2$. Note that this is a characterization *a posteriori* since the cost functional $J(x^0,u)$ depends on the final value $P_\beta(T)$.

5 Robustness of acceptable stability behaviour

Consider the uncertain system

$$\dot{x} = A_\Delta x := (A + B_1 \Delta C_1)x, \quad \Delta \in \mathbb{K}^{\ell\times q}, \|\Delta\| < \delta$$

where $(A, B_1, C_1) \in \mathbb{K}^{n\times n} \times \mathbb{K}^{n\times \ell} \times \mathbb{K}^{q\times n}$, $\sigma(A) \subset \mathbb{C}_-$ and $\delta > 0$. Assume that the nominal system $\dot{x} = Ax$ has a practical stability requirement in the sense that

$$\|e^{At}\| \leq Me^{\beta t}, \qquad t \geq 0, \quad (21)$$

where $M \geq 1$ and $\beta \leq 0$ are given. We now introduce a measure for the robustness of this condition under perturbations $A \rightsquigarrow A_\Delta$.

Definition 12. Given $M \geq 1$ and $\beta \leq 0$, the acceptability radius of A under perturbations of the form $A \rightsquigarrow A + B_1 \Delta C_1$, $\Delta \in \mathbb{K}^{\ell\times q}$ is defined by

$$r_{\mathbb{K}}(A; B_1, C_1; M, \beta) = \inf\{\|\Delta\|; \Delta \in \mathbb{K}^{\ell\times q}, \exists\, t \geq 0 : \|e^{(A+B_1\Delta C_1)t}\| > Me^{\beta t}\}.$$

It follows from this definition that every system $\dot{x} = A_\Delta x$ with $\|\Delta\| \leq r_{\mathbb{K}}$ has an acceptable stability behaviour.

Proposition 13. *Given $M \geq 1$, $\beta \leq 0$ suppose there exists $P^o \in \mathcal{H}_n$, $Q \in \mathcal{H}_q$, $R \in \mathcal{H}_\ell$, $P^o \succ 0$, $Q \succ 0$, $R \succ 0$ such that*

$$\dot{P} - A^*P - PA + 2\beta P - C_1^* Q C_1 - PB_1 R B_1^* P = 0, \quad P(0) = P^o \quad (22)$$

has a solution on $\mathbb{R}_+$ which satisfies

$$\overline{\sigma}(P(t))/\underline{\sigma}(P^o) \leq M^2, \quad t \geq 0. \quad (23)$$

Then $r_{\mathbb{K}}(A; B_1, C_1; M, \beta) \geq (\underline{\sigma}(Q)\,\underline{\sigma}(R))^{1/2}$.

The condition given in the above proposition can be effectively used even in the borderline case where $M = 1$ (in which case P^o must necessarily be a multiple of the identity matrix by (23)). This is illustrated in the following example.

Example 14. Suppose A is a normal matrix, $A = U^* \operatorname{diag}(\lambda_1, \lambda_2, \ldots, \lambda_n)\, U$, with U unitary, $\mathrm{Re}\,\lambda_i < 0$, $i \in \underline{n} := \{1, \ldots, n\}$ and $B_1 = C_1 = I_n$. Let

$P^o = I_n, Q = \alpha^2 I_n, R = \alpha^2 I_n$, then if we set $\hat{P}(t) = UP(t)U^* = \mathrm{diag}\,(p_1(t), p_2(t), \ldots, p_n(t))$, (22) is equivalent to the following set of n decoupled scalar differential Riccati equations:

$$\dot{p}_i - (\lambda_i + \lambda_i^* - 2\beta)p_i - \alpha^2 - \alpha^2 p_i^2 = 0, \;\; p_i(0) = 1, \; i \in \underline{n}.$$

Let $\gamma_i = -(\lambda_i + \lambda_i^* - 2\beta)/2, \quad \gamma_1 \leq \gamma_2 \leq \ldots \leq \gamma_n$ and suppose that β is such that $\gamma_1 > 0$. If $\alpha^2 = \gamma_1$, then $p_1(t) \equiv 1$ and $p_i(t) \leq 1$ for all $t \geq 0$ and $i \in \underline{n}$. So $1 \geq \|\hat{P}(t)\| = \|P(t)\|$, $t \geq 0$. Thus $r_{\mathbb{K}}(A; I, I; 1, \beta) \geq \gamma_1$. In particular $r_{\mathbb{K}}(A; I, I; 1, 0) \geq -1/2(\lambda_{\max}(A + A^*))$. It can be shown that this inequality is in fact an equality. □

Associated with the above stability requirement (21) is the following:

Robust Control Problem: *Suppose we are given a stabilizable control system of the form (4.13), structure matrices* $(B_1, C_1) \in \mathbb{K}^{n\times\ell} \times \mathbb{K}^{q\times n}$ *and* $M \geq 1$, $\beta \leq 0$. *For which* $\gamma > 0$ *does there exist a state feedback matrix* F *such that*

$$r_{\mathbb{K}}(A + B_2F; B_1, C_1; M, \beta) \geq \gamma\,?$$

Determine the supremum of these achievable robustness indices.

Note that for a given pair (A, B_2) the transient bound for any feedback system cannot be less than the transient bound of A^* restricted to the A^*-invariant uncontrollable subspace.
Finally we analyze the effect of nonlinear perturbations on the transient behaviour of a linear system. Consider the nonlinear equation

$$\dot{x} = Ax + B_1\Delta(C_1x), \quad x(0) = x^0, \qquad (24)$$

where $\Delta : \mathbb{K}^q \mapsto \mathbb{K}^\ell$ is locally Lipschitz and satisfies $\Delta(0) = 0$. The following proposition extends Proposition 13 to nonlinear perturbations of the form $A \rightsquigarrow A_\Delta$ where $A_\Delta(x) = Ax + B_1\Delta(C_1x)$.

Proposition 15. *Under the assumptions of Proposition 13 suppose that*

$$\|\Delta(x)\| \leq (\underline{\sigma}(Q)\underline{\sigma}(R))^{1/2}\|x\|, \quad x \in \mathbb{K}^q.$$

Then for every $x^0 \in \mathbb{K}^n$, *there exists a unique solution* $x(\cdot, x^0)$ *of* (24) *on* $\mathbb{R}_+$ *and*

$$\|x(t, x^0)\| \leq Me^{\beta t}\|x^0\|, \text{ for all } t \geq 0.$$

Example 16. Suppose that A is normal, $\sigma(A) \subset \mathbb{C}_-$, $\beta \leq 0$ and $\Delta : \mathbb{K}^n \mapsto \mathbb{K}^n$ is locally Lipschitz and satisfies $\Delta(0) = 0$. Then by using the same Riccati equation as in Example 14, we have that if $\lambda_{\max}(A + A^*) - 2\beta < 0$ and

$$\dot{x} = Ax + \Delta(x), \quad x(0) = x^0,$$
$$\|\Delta(x)\| \leq -(1/2)(\lambda_{\max}(A + A^*) - 2\beta)\|x\|,$$

the solutions $x(\cdot, x^0)$ satisfy $\|x(t, x^0)\| \leq e^{\beta t}\|x^0\|$, $t \geq 0$. □

References

[1] D. Hinrichsen and B. Kelb. "Stability radii and spectral value sets for real matrix perturbations", in U. Helmke, R. Mennicken, and J. Saurer (eds.), *Systems and Networks: Mathematical Theory and Applications. Volume II: Invited and Contributed Papers*, pages 217–220, Akademie-Verlag, Berlin, (1994)

[2] D. Hinrichsen and A. Pritchard, "Stability radius for structured perturbations and the algebraic Riccati equation", *Systems & Control Letters*, Vol.8, pp. 105–113, (1986)

[3] Hinrichsen, D. and Pritchard, A. J., "On the transient behaviour of stable linear systems", *Proceedings of MTNS2000*, Perpignan, (2000)

[4] Khusainov, D.Ya., Komarov, Yu.A., Yun'kova, Ye.A., "Constructing Optimal Lyapunov Functions for Linear Differential Equations", *Sov. Autom. Control*, Vol. 17(6), pp. 80–83, (1984)

[5] LeVeque, R.J., Trefethen, L.N., "On the resolvent condition in the Kreiss matrix theorem",*BIT*, Vol. 24, pp. 584–591, (1984)

[6] Malyshev, A.N. , Sadkane, M., "On the stability of large matrices", *Journal of Computational and Applied Mathematics*, Vol. 102, pp. 303–313, (1999)

[7] Ström, T., "On Logarithmic Norms", *SIAM J. Numer. Anal.*, Vol 12(5), pp. 741–753, (1975)

[8] Trefethen, L. N., Trefethen, A. E., Reddy, S. C., and Driscoll, T. A., "Hydrodynamic stability without eigenvalues", *Science*, Vol. 261, pp. 578–584, (1993)

[9] Trefethen, L. N., "Pseudospectra of linear operators", *SIAM Rev.*, Vol. 39, pp. 383–406, (1997)

[10] Wegert, E., Trefethen, L.N., "From the Buffon Needle Problem to the Kreiss Matrix Theorem", *Am. Math. Mon.*, Vol. 101(2), pp. 132–139, (1994)